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B. Sc (Part-1) Hons

State and prove Green's Theorem.

Green's Theorem:— The Theorem state that If ϕ and ψ are two Continously differentiable scalar functions such that $\nabla\phi$ and $\nabla\psi$ are also Continously differentiable, then

$$\iint_S (\phi\nabla\psi - \psi\nabla\phi)\,d\vec{s} = \iiint_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\,dv$$

proof!— Let a vector function $\vec{v}$ be given by $\vec{v} = \phi.\nabla\psi$ ——— (1)

where ϕ and ψ are scalar functions so that $\vec{v}$ is product of a scalar function ϕ and the gradient of another scalar function ψ.

Writing eqn (1) in terms of the Components,

$$v_x = \phi\frac{\partial\psi}{\partial x},\quad v_y = \phi.\frac{\partial\psi}{\partial y},\quad v_z = \phi.\frac{\partial\psi}{\partial z}$$

$$\therefore \nabla.\vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$= \frac{\partial}{\partial x}\left(\phi.\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\phi.\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\phi.\frac{\partial\psi}{\partial z}\right)$$

$$= \left[\phi.\frac{\partial^2\psi}{\partial x^2} + \frac{\partial\phi}{\partial x}.\frac{\partial\psi}{\partial x}\right] + \left[\phi.\frac{\partial^2\psi}{\partial y^2} + \frac{\partial\phi}{\partial y}.\frac{\partial\psi}{\partial y}\right]$$

$$+ \left[\phi.\frac{\partial^2\psi}{\partial z^2} + \frac{\partial\phi}{\partial z}.\frac{\partial\psi}{\partial z}\right]$$

$$= \left[\phi \cdot \frac{\partial^2 \psi}{\partial x^2} + \phi \cdot \frac{\partial^2 \psi}{\partial y^2} + \phi \cdot \frac{\partial^2 \psi}{\partial z^2}\right] + \left[\frac{\partial \phi}{\partial x} \cdot \frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y} \cdot \frac{\partial \psi}{\partial y} + \frac{\partial \phi}{\partial z} \cdot \frac{\partial \psi}{\partial z}\right]$$

$$= \phi \cdot \nabla^2 \psi + \nabla\phi \cdot \nabla\psi \quad \text{—— (2)}$$

By Gauss's theorem of divergence

$$\iint_S \vec{V} \cdot \vec{n}\, ds = \iiint_V (\nabla \times \vec{V}) \cdot dv \quad \text{—— (3)}$$

Hence substituting from eqnation (1), (2) and (3) we get

$$\iint_S (\phi \cdot \nabla\psi) \cdot \vec{n}\, ds = \iiint_V (\phi \cdot \nabla^2 \psi + \nabla\phi \cdot \nabla\psi)\, dv \quad \text{—— (4)}$$

If interchange ϕ and ψ in equation (4)

$$\iint_S (\psi \nabla\phi \cdot \vec{n}\, ds) = \iiint_V (\psi \nabla^2 \phi + \nabla\phi \cdot \nabla\psi)\, dv \quad \text{—— (5)}$$

Now subtracting eqn (5) from (4)

$$\iint_S (\phi \nabla\psi - \psi \nabla\phi)\, \vec{n} \cdot ds = \iiint_V (\phi \cdot \nabla^2 \psi - \psi \nabla^2 \phi)\, dv$$

or, $$\iint_S (\phi \cdot \nabla\psi - \psi \nabla\phi) \cdot d\vec{s} = \iiint_V (\phi \nabla^2 \psi + \psi \nabla^2 \phi)\, dv$$

This is Green's theorem